

# Generalized characteristics of the homogenous magneto hydrodynamical equations

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## Abstract

With the help of the generalized characteristics(GC) of the first order partial differential equations(PDE) we calculate the differential equation system of characteristics of the homogenous magneto hydrodynamical equations(MHD).

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## I. INTRODUCTION

The investigation of the magneto hydrodynamical(MHD) equations is crucial importance to understand plasma instabilities in the future fusion facilities like ITER. Nowadays large efforts are made to numerically solve the complete MHD equations on a torus or on a more realistic stellarator geometry. [1] However, simplified models of the complete MHD equation may help us to get a deeper understanding physical mechanism of plasma or let more insight into the structure of magnetized fluids. In the following study we shortly introduce the mathematical formalism of the generalized characteristics of first order partial differential equations and apply this theory to the full three dimensional MHD equations.

## II. THEORY

It is well known from the theory of first order linear PDEs that along the characteristic line the original PDE becomes an ordinary differential equation(ODE). Therefore, we can make an important observation, along the characteristics the solution is constant. For linear equations the characteristic curve is a line. For one space and time dimension the characteristic lines give us a qualitative picture about the solution of the equation on the plain. The proof of this statement can be found in any textbook [2]

In the following we briefly introduce the mathematics of the generalized characteristics of the first order PDEs. Let's consider the following first order PDE, which can be non-linear as well:

$$F(x, t, u(x, t), p, q) = 0 \quad (1)$$

we use the standard notation of  $p = \partial u(x, t)/\partial x$ ,  $q = \partial u(x, t)/\partial t$ . According to the book of Melikyan [4] the differential equation system of the characteristics is the following:

$$\dot{x} = F_p, \quad \dot{t} = F_q, \quad (2)$$

$$\dot{u}(x, t) = p \cdot F_p + q \cdot F_q, \quad (3)$$

$$\dot{p} = -F_x + p \cdot F_u, \quad \dot{q} = -F_t + q \cdot F_u \quad (4)$$

where  $\dot{\bullet} = \frac{d}{d\tau}$ . To avoid further confusions and misunderstanding we use the more detailed notation:

$$\frac{dx}{d\tau} = \frac{\partial F}{\partial(\frac{\partial u}{\partial x})}, \quad \frac{dt}{d\tau} = \frac{\partial F}{\partial(\frac{\partial u}{\partial t})}, \quad (5)$$

$$\frac{du}{d\tau} = \frac{\partial u}{\partial x} \cdot \frac{\partial F}{\partial(\frac{\partial u}{\partial x})} + \frac{\partial u}{\partial t} \cdot \frac{\partial F}{\partial(\frac{\partial u}{\partial t})}, \quad (6)$$

$$\frac{dp}{d\tau} = -\frac{\partial F}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial F}{\partial u}, \quad \frac{dq}{d\tau} = -\frac{\partial F}{\partial t} - \frac{\partial u}{\partial t} \cdot \frac{\partial F}{\partial u} \quad (7)$$

$$(8)$$

These equations can be easily generalized to a higher dimensional equation system  $i = 1 \dots n$  as well.

$$F_i(x, y, z, t, u_i(x, y, z, t), p_i, q_i, r_i, s_i) = 0 \quad (9)$$

with the notation of  $p_i = \partial u_i(x, y, z, t)/\partial x$ ,  $q_i = \partial u_i(x, y, z, t)/\partial y$ ,  $r_i = \partial u_i(x, y, z, t)/\partial z$ ,  $s_i = \partial u_i(x, y, z, t)/\partial t$ . with the above given detailed notation

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{\partial F_i}{\partial(\frac{\partial u_i}{\partial x})} \\ \frac{dy}{d\tau} &= \frac{\partial F_i}{\partial(\frac{\partial u_i}{\partial y})} \\ \frac{dz}{d\tau} &= \frac{\partial F_i}{\partial(\frac{\partial u_i}{\partial z})} \\ \frac{dt}{d\tau} &= \frac{\partial F_i}{\partial(\frac{\partial u_i}{\partial t})} \\ \frac{du_i}{d\tau} &= \frac{\partial u_i}{\partial x} \cdot \frac{\partial F_i}{\partial(\frac{\partial u_i}{\partial x})} + \frac{\partial u_i}{\partial y} \cdot \frac{\partial F_i}{\partial(\frac{\partial u_i}{\partial y})} + \frac{\partial u_i}{\partial z} \cdot \frac{\partial F_i}{\partial(\frac{\partial u_i}{\partial z})} + \frac{\partial u_i}{\partial t} \cdot \frac{\partial F_i}{\partial(\frac{\partial u_i}{\partial t})} \\ \frac{dp_i}{d\tau} &= -\frac{\partial F_i}{\partial x} - \frac{\partial u_i}{\partial x} \cdot \frac{\partial F_i}{\partial u_i} \\ \frac{dq_i}{d\tau} &= -\frac{\partial F_i}{\partial y} - \frac{\partial u_i}{\partial y} \cdot \frac{\partial F_i}{\partial u_i} \\ \frac{dr_i}{d\tau} &= -\frac{\partial F_i}{\partial z} - \frac{\partial u_i}{\partial z} \cdot \frac{\partial F_i}{\partial u_i} \\ \frac{ds_i}{d\tau} &= -\frac{\partial F_i}{\partial t} - \frac{\partial u_i}{\partial t} \cdot \frac{\partial F_i}{\partial u_i} \end{aligned} \quad (10)$$

These kind of equations can be applied to any kind of hyperbolic equation systems, like gas dynamical problems [2], multi-phase flows [3] or to MHD.

The governing equations for an ideal, non-relativistic compressible plasma may be written in different forms if the following assumptions hold:

$$\frac{\lambda}{L} \ll 1, \quad \frac{\epsilon}{\tau\sigma} \ll 1, \quad \left(\frac{v}{c}\right)^2 \ll 1, \quad \frac{\mu}{\rho V L} \ll 1 \quad (11)$$

where  $\rho, v, \tau$  and  $L$  are, respectively, characteristic density, speed, time and length scales for the problem,  $c$  is the speed of light, and  $\epsilon$  and  $\sigma$  is the dielectric constant and conductivity of the fluid. In conservative variables, the governing equations, which is a combination of Euler equations of gas dynamics and the Maxwell equations of electromagnetics, is the following:

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ \mathbf{B} \\ E \end{pmatrix} + \nabla \cdot \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \mathbf{v} + \mathbf{I} \left( \pi + \frac{\mathbf{B} \mathbf{B}}{2} \right) - \mathbf{B} \mathbf{B} \\ \mathbf{v} \mathbf{B} - \mathbf{B} \mathbf{v} \\ (E + \pi + \frac{\mathbf{B} \mathbf{B}}{2}) \mathbf{v} - \mathbf{B} (\mathbf{v} \cdot \mathbf{B}) \end{pmatrix} = 0 \quad (12)$$

where  $\mathbf{I}$  is the  $3 \times 3$  identity matrix,  $\rho$  is the density,  $\mathbf{v}$  is the velocity,  $\pi$  is the pressure (to avoid confusion with the notation used for partial differential equations  $p = \partial u(x, t) / \partial x$ ),  $\mathbf{B}$  is the magnetic field, and  $E$  is the energy, defined as:

$$E = \frac{\pi}{\gamma - 1} + \rho \frac{\mathbf{v} \cdot \mathbf{v}}{2} + \frac{\mathbf{B} \cdot \mathbf{B}}{2} \quad (13)$$

Solution of these equations can help to understand a number of problems governed by fluid-dynamics and electromagnetic effects.

Given the following primitive variables  $\mathbf{w} = (\rho, v_x, v_y, v_z, B_x, B_y, B_z, \pi)$ , the MHD equations (12) may be written in quasi-linear form as:

$$\frac{\partial \mathbf{w}}{\partial t} + \underline{\underline{A}} \frac{\partial \mathbf{w}}{\partial x} + \underline{\underline{B}} \frac{\partial \mathbf{w}}{\partial y} + \underline{\underline{C}} \frac{\partial \mathbf{w}}{\partial z} = 0 \quad (14)$$

where  $\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}}$  are  $8 \times 8$  matrices.

With a

$$\underline{\underline{A}} = \begin{bmatrix} v_x & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_x & 0 & 0 & -\frac{B_x}{\rho} & \frac{B_y}{\rho} & \frac{B_z}{\rho} & \frac{1}{\rho} \\ 0 & 0 & v_x & 0 & -\frac{B_y}{\rho} & -\frac{B_x}{\rho} & 0 & 0 \\ 0 & 0 & 0 & v_x & -\frac{B_z}{\rho} & 0 & -\frac{B_x}{\rho} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_y & -B_x & 0 & -v_y & v_x & 0 & 0 \\ 0 & B_z & 0 & -B_x & -v_z & 0 & v_x & 0 \\ 0 & \gamma \pi & 0 & 0 & (\gamma - 1) \mathbf{v} \cdot \mathbf{B} & 0 & 0 & v_x \end{bmatrix} \quad (15)$$

where  $\gamma$  is the compressibility of the fluid. Matrix  $\underline{\underline{A}}$  is singular - the fifth is zero, leading to a zero eigenvalue after the diagonalization.

$$\lambda_1^A = 0, \quad \lambda_2^A = v_x, \quad \lambda_3^A = v_x + \frac{B_x}{\sqrt{\rho}}, \quad \lambda_4^A = v_x - \frac{B_x}{\sqrt{\rho}}$$

$$\lambda_5^A = v_x + \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi + \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_x^2}} \quad (16)$$

$$\lambda_6^A = v_x - \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi + \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_x^2}} \quad (17)$$

$$\lambda_7^A = v_x + \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi - \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_x^2}} \quad (18)$$

$$\lambda_8^A = v_x - \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi - \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_x^2}} \quad (19)$$

The eigenvalues of the system are well known, and they correspond to:

- one entropy wave  $\lambda_2^A$  traveling with speed  $v_x$ ,
- two Alfvén waves  $\lambda_{2,3}^A$  traveling with speed  $v_x \pm c_c$  where  $c_a = \frac{B_x}{\sqrt{\rho}}$  is the Alfvén speed,
- four magneto-acoustic waves (  $\lambda_5^A \dots \lambda_8^A$  )

For completeness we calculate and present the eigenvalues for the y and z direction too. Matrices  $\underline{\underline{B}}, \underline{\underline{C}}$  and the corresponding eigenvalues are very similar to  $\underline{\underline{A}}$  and have the same structure.

$$\underline{\underline{B}} = \begin{bmatrix} v_y & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & v_y & 0 & 0 & -\frac{B_y}{\rho} & -\frac{B_x}{\rho} & 0 & 0 \\ 0 & 0 & v_y & 0 & \frac{B_x}{\rho} & -\frac{B_y}{\rho} & \frac{B_z}{\rho} & \frac{1}{\rho} \\ 0 & 0 & 0 & v_y & 0 & -\frac{B_z}{\rho} & -\frac{B_y}{\rho} & 0 \\ 0 & -B_y & -B_x & 0 & v_y & -v_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_z & -B_y & 0 & v_z & v_y & 0 \\ 0 & 0 & \gamma\pi & 0 & 0 & (\gamma - 1)\mathbf{v} \cdot \mathbf{B} & 0 & v_y \end{bmatrix} \quad (20)$$

$$\lambda_1^B = 0, \quad \lambda_2^B = v_y, \quad \lambda_3^B = v_y + \frac{B_y}{\sqrt{\rho}}, \quad \lambda_4^B = v_y - \frac{B_y}{\sqrt{\rho}}$$

$$\lambda_5^B = v_y + \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi + \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_y^2}} \quad (21)$$

$$\lambda_6^B = v_y - \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi + \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_y^2}} \quad (22)$$

$$\lambda_7^B = v_y + \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi - \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_y^2}} \quad (23)$$

$$\lambda_8^B = v_y - \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi - \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_y^2}} \quad (24)$$

$$\underline{\underline{C}} = \begin{bmatrix} v_z & 0 & 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & v_z & 0 & 0 & -\frac{B_z}{\rho} & 0 & -\frac{B_x}{\rho} & 0 \\ 0 & 0 & v_z & 0 & 0 & -\frac{B_z}{\rho} & -\frac{B_y}{\rho} & 0 \\ 0 & 0 & 0 & v_z & \frac{B_x}{\rho} & \frac{B_y}{\rho} & -\frac{B_z}{\rho} & \frac{1}{\rho} \\ 0 & -B_z & 0 & B_x & v_z & 0 & -v_x & 0 \\ 0 & 0 & -B_z & B_y & 0 & v_z & -v_y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma\pi & 0 & 0 & (g-1)\mathbf{v}\mathbf{B} & v_z \end{bmatrix} \quad (25)$$

$$\lambda_1^C = 0, \quad \lambda_2^C = v_z, \quad \lambda_3^C = v_z + \frac{B_z}{\sqrt{\rho}}, \quad \lambda_4^C = v_z - \frac{B_z}{\sqrt{\rho}}$$

$$\lambda_5^C = v_z + \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi + \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_z^2}} \quad (26)$$

$$\lambda_6^C = v_z - \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi + \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_z^2}} \quad (27)$$

$$\lambda_7^C = v_z + \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi - \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_z^2}} \quad (28)$$

$$\lambda_8^C = v_z - \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi - \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_z^2}} \quad (29)$$

The matrix form of the MHD equations (14) reads as follows:

$$\begin{aligned}
& \frac{\partial \rho}{\partial t} = 0 \\
& \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} = 0 \\
& \frac{\partial v_y}{\partial t} + \left( v_x + \frac{B_x}{\sqrt{\rho}} \right) \frac{\partial v_y}{\partial x} + \left( v_y + \frac{B_y}{\sqrt{\rho}} \right) \frac{\partial v_y}{\partial y} + \left( v_z + \frac{B_z}{\sqrt{\rho}} \right) \frac{\partial v_y}{\partial z} = 0 \\
& \frac{\partial v_z}{\partial t} + \left( v_x - \frac{B_x}{\sqrt{\rho}} \right) \frac{\partial v_z}{\partial x} + \left( v_y - \frac{B_y}{\sqrt{\rho}} \right) \frac{\partial v_z}{\partial y} + \left( v_z - \frac{B_z}{\sqrt{\rho}} \right) \frac{\partial v_z}{\partial z} = 0 \\
& \frac{\partial B_x}{\partial t} + \left( v_x + \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi + \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_x^2}} \right) \frac{\partial B_x}{\partial x} + \\
& \quad \left( v_y + \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi + \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_y^2}} \right) \frac{\partial B_x}{\partial y} + \\
& \quad \left( v_z + \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi + \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_z^2}} \right) \frac{\partial B_x}{\partial z} = 0 \\
& \frac{\partial B_y}{\partial t} + \left( v_x - \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi + \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_x^2}} \right) \frac{\partial B_y}{\partial x} + \\
& \quad \left( v_y - \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi + \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_y^2}} \right) \frac{\partial B_y}{\partial y} + \\
& \quad \left( v_z - \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi + \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_z^2}} \right) \frac{\partial B_y}{\partial z} = 0 \\
& \frac{\partial B_z}{\partial t} + \left( v_x + \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi - \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_x^2}} \right) \frac{\partial B_z}{\partial x} + \\
& \quad \left( v_y + \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi - \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_y^2}} \right) \frac{\partial B_z}{\partial y} + \\
& \quad \left( v_z + \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi - \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_z^2}} \right) \frac{\partial B_z}{\partial z} = 0 \\
& \frac{\partial \pi}{\partial t} + \left( v_x - \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi - \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_x^2}} \right) \frac{\partial \pi}{\partial x} + \\
& \quad \left( v_y - \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi - \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_y^2}} \right) \frac{\partial \pi}{\partial y} + \\
& \quad \left( v_z - \frac{1}{\sqrt{2\rho}} \sqrt{\mathbf{B}^2 + \gamma\pi - \sqrt{(\mathbf{B}^2 + \gamma\pi)^2 - 4\gamma\pi B_z^2}} \right) \frac{\partial \pi}{\partial z} = 0
\end{aligned} \tag{30}$$

Now applying the equations of (10), after a tedious derivation we may get the complete equation system of the generalized characteristics.

For the first equation of (30)  $\frac{\partial \rho}{\partial t} = 0$  the equation system of characteristics is trivial  $\frac{dt}{d\tau} = 1$ ,  $\frac{d\rho}{d\tau} = s$  where  $\tau$  is the parameter of the characteristic curve. It is clear that time can be used as a natural parameter too, so the equation of the first characteristics is  $\frac{d\rho}{dt} = s_1$ .

For the second equation of (30) the system becomes much more complicated:

$$\begin{aligned}\frac{dx}{dt} &= v_x, \quad \frac{dy}{dt} = v_y, \quad \frac{dz}{dt} = v_z, \quad , \quad \frac{dv_x}{dt} = p_2 v_x + q_2 v_y + r_2 v_z + s_2, \quad \frac{dp_2}{dt} = -p_2^2, \quad , \quad \frac{dq_2}{dt} = -p_2 q_2, \\ \frac{dr_2}{dt} &= -p_2 r_2, \quad \frac{ds_2}{dt} = -p_2 s_2,\end{aligned}$$

The equation systems of the last six variable are the following:

$$\begin{aligned}\frac{dx}{dt} &= v_x + \frac{B_x}{\sqrt{\rho}}, \quad \frac{dy}{dt} = v_y + \frac{B_y}{\sqrt{\rho}}, \quad \frac{dz}{dt} = v_z + \frac{B_z}{\sqrt{\rho}}, \quad \frac{dv_y}{dt} = p_3(v_x + \frac{B_x}{\sqrt{\rho}}) + q_3(v_y + \frac{B_y}{\sqrt{\rho}}) + r_3(v_z + \frac{B_z}{\sqrt{\rho}}) + s_3, \\ \frac{dp_3}{dt} &= -p_3 q_3, \quad \frac{dq_3}{dt} = -q_3^2, \quad \frac{dr_3}{dt} = -r_3 q_3, \quad \frac{ds_3}{dt} = -p_3 s_3,\end{aligned}$$

$$\begin{aligned}\frac{dx}{dt} &= v_x - \frac{B_x}{\sqrt{\rho}}, \quad \frac{dy}{dt} = v_y - \frac{B_y}{\sqrt{\rho}}, \quad \frac{dz}{dt} = v_z - \frac{B_z}{\sqrt{\rho}}, \quad \frac{dv_z}{dt} = p_4(v_x - \frac{B_x}{\sqrt{\rho}}) + q_4(v_y - \frac{B_y}{\sqrt{\rho}}) + r_4(v_z - \frac{B_z}{\sqrt{\rho}}) + s_4, \\ \frac{dp_4}{dt} &= -p_4 r_4, \quad \frac{dq_4}{dt} = -q_4 r_4, \quad \frac{dr_4}{dt} = -r_4^2, \quad \frac{ds_4}{dt} = -r_4 s_4,\end{aligned}$$

$$\begin{aligned}\frac{dx}{dt} &= \lambda_5^A, \quad \frac{dy}{dt} = \lambda_5^B, \quad \frac{dz}{dt} = \lambda_5^C, \quad \frac{dB_x}{dt} = p_5 \lambda_5^A + q_5 \lambda_5^B + r_5 \lambda_5^C + s_5, \\ \frac{dp_5}{dt} &= -p_5 \left[ \left( \frac{\partial}{\partial B_x} \lambda_5^A \right) p_5 + \left( \frac{\partial}{\partial B_x} \lambda_5^B \right) q_5 + \left( \frac{\partial}{\partial B_x} \lambda_5^C \right) r_5 \right], \\ \frac{dq_5}{dt} &= -q_5 \left[ \left( \frac{\partial}{\partial B_x} \lambda_5^A \right) p_5 + \left( \frac{\partial}{\partial B_x} \lambda_5^B \right) q_5 + \left( \frac{\partial}{\partial B_x} \lambda_5^C \right) r_5 \right], \\ \frac{dr_5}{dt} &= -r_5 \left[ \left( \frac{\partial}{\partial B_x} \lambda_5^A \right) p_5 + \left( \frac{\partial}{\partial B_x} \lambda_5^B \right) q_5 + \left( \frac{\partial}{\partial B_x} \lambda_5^C \right) r_5 \right], \\ \frac{ds_5}{dt} &= -s_5 \left[ \left( \frac{\partial}{\partial B_x} \lambda_5^A \right) p_5 + \left( \frac{\partial}{\partial B_x} \lambda_5^B \right) q_5 + \left( \frac{\partial}{\partial B_x} \lambda_5^C \right) r_5 \right],\end{aligned}$$

$$\begin{aligned}\frac{dx}{dt} &= \lambda_6^A, \quad \frac{dy}{dt} = \lambda_6^B, \quad \frac{dz}{dt} = \lambda_6^C, \quad \frac{dB_y}{dt} = p_6 \lambda_6^A + q_6 \lambda_6^B + r_6 \lambda_6^C + s_6, \\ \frac{dp_6}{dt} &= -p_6 \left[ \left( \frac{\partial}{\partial B_y} \lambda_6^A \right) p_6 + \left( \frac{\partial}{\partial B_y} \lambda_6^B \right) q_6 + \left( \frac{\partial}{\partial B_y} \lambda_6^C \right) r_6 \right], \\ \frac{dq_6}{dt} &= -q_6 \left[ \left( \frac{\partial}{\partial B_y} \lambda_6^A \right) p_6 + \left( \frac{\partial}{\partial B_y} \lambda_6^B \right) q_6 + \left( \frac{\partial}{\partial B_y} \lambda_6^C \right) r_6 \right], \\ \frac{dr_6}{dt} &= -r_6 \left[ \left( \frac{\partial}{\partial B_y} \lambda_6^A \right) p_6 + \left( \frac{\partial}{\partial B_y} \lambda_6^B \right) q_6 + \left( \frac{\partial}{\partial B_y} \lambda_6^C \right) r_6 \right], \\ \frac{ds_6}{dt} &= -s_6 \left[ \left( \frac{\partial}{\partial B_y} \lambda_6^A \right) p_6 + \left( \frac{\partial}{\partial B_y} \lambda_6^B \right) q_6 + \left( \frac{\partial}{\partial B_y} \lambda_6^C \right) r_6 \right],\end{aligned}$$

$$\begin{aligned}\frac{dx}{dt} &= \lambda_7^A, \quad \frac{dy}{dt} = \lambda_7^B, \quad \frac{dz}{dt} = \lambda_7^C, \quad \frac{dB_z}{dt} = p_7 \lambda_7^A + q_7 \lambda_7^B + r_7 \lambda_7^C + s_7, \\ \frac{dp_7}{dt} &= -p_7 \left[ \left( \frac{\partial}{\partial B_z} \lambda_7^A \right) p_7 + \left( \frac{\partial}{\partial B_z} \lambda_7^B \right) q_7 + \left( \frac{\partial}{\partial B_z} \lambda_7^C \right) r_7 \right], \\ \frac{dq_7}{dt} &= -q_7 \left[ \left( \frac{\partial}{\partial B_z} \lambda_7^A \right) p_7 + \left( \frac{\partial}{\partial B_z} \lambda_7^B \right) q_7 + \left( \frac{\partial}{\partial B_z} \lambda_7^C \right) r_7 \right], \\ \frac{dr_7}{dt} &= -r_7 \left[ \left( \frac{\partial}{\partial B_z} \lambda_7^A \right) p_7 + \left( \frac{\partial}{\partial B_z} \lambda_7^B \right) q_7 + \left( \frac{\partial}{\partial B_z} \lambda_7^C \right) r_7 \right], \\ \frac{ds_7}{dt} &= -s_7 \left[ \left( \frac{\partial}{\partial B_z} \lambda_7^A \right) p_7 + \left( \frac{\partial}{\partial B_z} \lambda_7^B \right) q_7 + \left( \frac{\partial}{\partial B_z} \lambda_7^C \right) r_7 \right],\end{aligned}$$

$$\frac{dx}{dt} = \lambda_8^A, \quad \frac{dy}{dt} = \lambda_8^B, \quad \frac{dz}{dt} = \lambda_8^C, \quad \frac{d\pi}{dt} = p_8 \lambda_8^A + q_8 \lambda_8^B + r_8 \lambda_8^C + s_8,$$



$$\begin{aligned}
\frac{dp_8}{dt} &= -p_8 \left[ \left( \frac{\partial}{\partial \pi} \lambda_8^A \right) p_8 + \left( \frac{\partial}{\partial \pi} \lambda_8^B \right) q_8 + \left( \frac{\partial}{\partial \pi} \lambda_8^C \right) r_8 \right], \\
\frac{dq_8}{dt} &= -q_8 \left[ \left( \frac{\partial}{\partial \pi} \lambda_8^A \right) p_8 + \left( \frac{\partial}{\partial \pi} \lambda_8^B \right) q_8 + \left( \frac{\partial}{\partial \pi} \lambda_8^C \right) r_8 \right], \\
\frac{dr_8}{dt} &= -r_8 \left[ \left( \frac{\partial}{\partial \pi} \lambda_8^A \right) p_8 + \left( \frac{\partial}{\partial \pi} \lambda_8^B \right) q_8 + \left( \frac{\partial}{\partial \pi} \lambda_8^C \right) r_8 \right], \\
\frac{ds_8}{dt} &= -s_8 \left[ \left( \frac{\partial}{\partial \pi} \lambda_8^A \right) p_8 + \left( \frac{\partial}{\partial \pi} \lambda_8^B \right) q_8 + \left( \frac{\partial}{\partial \pi} \lambda_8^C \right) r_8 \right],
\end{aligned}$$

to avoid too lengthy formulas we used the standard eigenvalue notations  $(\lambda_k^j, j = A, B, C, k = 5, 6, 7, 8)$  for the last four magneto-acoustic waves. The complete system of equations for the characteristic curves are  $8 \times 8 = 64$  equations, unfortunately the author has problems to interpret the results in the present form. We can restrict the motion to reduce the complexity of the dynamics. The simplest case is a one dimensional motion in a one dimensional magnetic field which means a system with 4 PDEs. A possible stability analysis of a restricted motion may give new insight into the dynamics of the MHD equations. We can imagine that even more complex analysis like the Panlevé test or some kind of stability analysis can be applied to the presented differential equation system, unfortunately such investigation are out of the skill of the author. We can only hope that our work may give some impetus and motivate some reader to start such studies and investigate the equations presented above.

### III. SUMMARY

We briefly presented how the generalized characteristics of the homogenous magneto hydrodynamical equations can be derived. Unfortunately, no deeper e.g. geometrical interpretation of the equation could be found, other detailed analysis is also lacking or may remain to the reader.

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